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Homogeneous Field and WKB Approximation In Deformed Quantum Mechanics with Minimal Length

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Abstract

In the framework of the deformed quantum mechanics with minimal length, we consider the motion of a non-relativistic particle in a homogeneous external field. We find the integral representation for the physically acceptable wave function in the position representation. Using the method of steepest descent, we obtain the asymptotic expansions of the wave function at large positive and negative arguments. We then employ the leading asymptotic expressions to derive the WKB connection formula, which proceeds from classically forbidden region to classically allowed one through a turning point. By the WKB connection formula, we prove the Bohr-Sommerfeld quantization rule up to $\mathcal{O}(\beta)$. We also show that, if the slope of the potential at a turning point is too steep, the WKB connection formula fall apart around the turning point.

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I. INTRODUCTION

One of the predictions shared by various quantum theories of gravity is the existence of an effective cutoff in the ultraviolet, i.e., a minimal observable length[1–6]. Some realizations of the minimal length from various scenarios are proposed. One of the most popular models is the generalized uncertainty principle (GUP), derived from the modified fundamental commutation relation [7–14]

$$[X, P] = i\hbar(1 + \beta P^2), \tag{1}$$

where $\beta = \beta_0 \ell_p^2/\hbar^2 = \beta_0/c^2 M_p^2$ with the Planck mass $M_p = \sqrt{\hbar c/G}$ and the Planck length $\ell_p = \sqrt{G\hbar/c^3}$. β_0 is a dimensionless parameter. With this generalization, one can easily

derive the generalized uncertainty principle (GUP)

$$\Delta X \Delta P \geq \frac{\hbar}{2}[1+\beta(\Delta P)^2]. \tag{2}$$

This in turn gives the minimal measurable length

$$\Delta X \geq \Delta_{\min} = \hbar\sqrt{\beta} = \sqrt{\beta_0}\ell_p. \tag{3}$$

Eqn. (1) is the simplest model where only the minimal uncertainty in position is taken into account while the momentum can be infinite.

In this paper we consider one dimensional quantum mechanics with the deformed commutation relation (1). To implement the deformed commutators (1), one defines[12, 15]

$$X = X_0,\ P = P_0\left(1+\frac{\beta}{3}P_0^2\right), \tag{4}$$

where $[X_0, P_0] = i\hbar$, the usual canonical operators. One can easily show that to the first order of β, eqn. (1) is guaranteed. Henceforth, terms of $\mathcal{O}\left(\beta^2\right)$ and higher are neglected in the remainder of the paper. For a quantum system described by

$$H = \frac{P^2}{2m} + V\left(X\right), \tag{5}$$

the Hamiltonians can be written as

$$H = H_0 + H_1 + \mathcal{O}\left(\beta^2\right), \tag{6}$$

where $H_0 = \frac{P_0^2}{2m} + V\left(X_0\right)$ and $H_1 = \frac{2\beta}{3}P_0^2$. Furthermore, one can adopt the momentum representation

$$X_0 = i\hbar\frac{\partial}{\partial p},\ P_0 = p, \tag{7}$$

or the position representation

$$X_0 = x,\ P_0 = \frac{\hbar}{i}\frac{\partial}{\partial x}. \tag{8}$$

The momentum representation is very handy in the discussions of certain problems, such as the harmonic oscillator[16], the Coulomb potential[17, 18] and the gravitational well[19, 20]. Recently, a wide class of problems, like scattering from a barrier or a particle in a square well[21–23] are discussed in position representation. Moreover, in the position representation, it is much easier to derive and discuss WKB approximation in the deformed quantum mechanics analogously to in the ordinary quantum mechanics[24]. Thus, we adopt

the position representation in this paper. In the position representation, the deformed stationary Schrodinger equation is

$$\frac{d^2\psi(x)}{dx^2} - \ell_\beta^2 \frac{d^4\psi(x)}{dx^4} + \frac{2m(E - V(x))}{\hbar^2}\psi(x) = 0. \tag{9}$$

where we define $\ell_\beta^2 = \frac{2}{3}\hbar^2\beta^2$ for later convenience.

Although, the homogeneous field potential $V(X) = FX$ is not studied so intensively as the quantum well, it has an important application in theoretical physics. In the ordinary quantum mechanics, the solutions to the Schrodinger equation with the linear potential are Airy functions, which are essential to derive the WKB connection formulas through a turning point. This motivates us to study the linear potential in the deformed quantum mechanics.

In the deformed quantum mechanics with minimal length, the WKB approximation formulas are obtained in [24]. In addition, the deformed Bohr-Sommerfeld quantization is used to acquire energy spectra of bound states in various potentials[18, 23–26]. Therefore, it is interesting to derive the WKB connection formulas through a turning point and rigorously verify the Bohr-Sommerfeld quantization rule claimed before, which are presented in our paper. Besides, we find that, if the slope of the potential is too steep at a turning point, the WKB connection algorithm fails around the turning point. This is not unexpected because, if one makes linear approximation to the potential around such a turning point for asymptotic matching, the corrections to the wave functions due to the Hamiltonian H_1 become dominant before one reaches the WKB valid region.

This paper is organized as follows: In section II we give the integral representation of the physically acceptable wave function of the homogeneous field and its leading asymptotic behavior at large positive value of ρ. In section III, we obtain the asymptotic expansions of the physically acceptable wave function at both large positive and large negative values of ρ. Section IV is devoted to deriving the WKB connection formula and the related discussions. In section V, we offer a summary and conclusion.

II. DEFORMED SCHRODINGER EQUATION

Let us consider one-dimensional motion of a particle in a homogenous field, specifically in a field with the potential $V(X) = FX$. Here we take the direction of the force along the

axis of $-x$ and let F be the force exerting on the particle in the field. As discussed in the introduction, the deformed Schrodinger equation for this scenario is

$$\frac{d^2\psi(x)}{dx^2} - \ell_\beta^2 \frac{d^4\psi(x)}{dx^4} + \frac{2m(E-Fx)}{\hbar^2}\psi(x) = 0. \tag{10}$$

In order to solve eqn. (10), a new dimensionless variable ρ is introduced as

$$\rho = \left(x - \frac{E}{F}\right)\left(2mF/\hbar^2\right)^{\frac{1}{3}}. \tag{11}$$

Eqn. (10) then becomes

$$-\alpha^2\psi^{(4)} + \psi'' - \rho\psi = 0. \tag{12}$$

where we define another dimensionless variable $\alpha^2 = \ell_\beta^2 \left(2mF/\hbar^2\right)^{\frac{2}{3}}$ and the derivatives are in terms of the new variable ρ. The linear differential equation (12) is quartic and then there are four linearly independent solutions. We will shortly show that only one of them is physically acceptable.

A. Physically Acceptable Solution

The condition $\beta P^2 \ll 1$ validating our effective GUP model implies

$$\beta \langle x| P^2 |\psi\rangle \ll \langle x|\psi\rangle \Longrightarrow \alpha^2 |\psi''(\rho)| \ll |\psi(\rho)|. \tag{13}$$

This condition is also expected in the momentum space. Since the GUP model is only valid below the energy scale $\beta^{-\frac{1}{2}}$, the momentum spectrum of the state $|\psi\rangle$ should be greatly suppressed around the scale $\beta^{-\frac{1}{2}}$. It also leads to the condition (13). Moreover, the condition (13) and eqn. (12) give

$$\alpha^2 |\rho| \ll 1. \tag{14}$$

In other words, our GUP model, which is an effective model, is valid only when the condition (14) holds. Considering that the Compton wavelength of a particle should be much larger than $\hbar\sqrt{\beta}$ or ℓ_β in the GUP model, one can also obtain the condition (14) in the classical allowed region where $\rho < 0$. In a field with the potential $V(x)$, the kinematics energy of a non-relativistic particle is $E - V(x)$ and its momentum is $\sqrt{2m(E-V(x))}$. Therefore, the fact that the Compton wavelength of the particle $\lambda_c = \frac{\hbar}{\sqrt{2m(E-V(x))}}$ is much larger than ℓ_β yields $\alpha^2|\rho| \ll 1$. In the remainder of our paper except subsubsection IV B 1, we assume

$\alpha \ll 1$ which is useful to derive WKB connection formula around a smooth tuning point. One needs to consider $\alpha \gtrapprox 1$ scenario only when it comes to the WKB connection around a sharp turning point.

We notice that $E < V$ for $\rho > 0$. The wave function ψ is then exponentially damped for large positive value of ρ. Thus, one needs to evaluate asymptotic values of $\psi(\rho)$ at large positive value of ρ to find physically acceptable solution to eqn. (12). Note that, only when $\alpha \ll 1$, one can analyze asymptotic behavior of $\psi(\rho)$ at large positive value of ρ in the physically acceptable region where $\alpha^2 |\rho| \ll 1$. To determine the leading behavior of $\psi(\rho)$ at large positive value of ρ, we make the exponential substitution $\psi(\rho) = e^{s(\rho)}$ and then obtain for eqn. (12)

$$s'' + s'^2 - \rho - \alpha^2 \left[s^{(4)} + 6s'^2 s'' + 3s''^2 + 4s' s^{(3)} + s'^4 \right] = 0. \tag{15}$$

Eqn. (15) is as difficult to solve as eqn. (12). Here our strategy to find the asymptotic behavior of $\psi(\rho)$ from eqn. (15) is as follows[27]:

(a) We neglect all terms appearing small and approximate the exact differential equation with the asymptotic one.

(b) We solve the resulting equation and check that the solution is consistent with approximations made in step (a).

It is usually true that higher derivative terms than s' are discarded in step (a). Therefore, we reduce eqn. (15) to the asymptotic differential equation

$$s'^2 - \alpha^2 s'^4 \sim \rho. \tag{16}$$

Solving eqn. (16) gives four solutions for s', two of which are discarded considering $\beta P^2 \ll 1$. Taking asymptotic relation (13) into account, one can further reduce the quartic equation (16) to a quadratic equation

$$s'^2 \sim \rho. \tag{17}$$

which has only two solutions for s'. The two solutions are $s' \sim \pm\sqrt{\rho}$, and, therefore,

$$\psi(\rho) \sim \exp\left(\pm \frac{2}{3} \rho^{\frac{3}{2}} \right), \text{ at large positive value of } \rho, \tag{18}$$

where $-$ is for the physically acceptable solution. It is easy to check that the solution $s' \sim \pm\sqrt{\rho}$ satisfy the assumptions

$$s'', \, s'^2 s'', \, s^{(3)}, \, s''^2, \, s' s^{(3)} \text{ and } s^{(4)} \ll \rho,$$

as long as $\rho \gg 1$.

It is interesting to note that the two discarded solutions of eqn. (16) are

$$s' \sim \pm \frac{\sqrt{1+\sqrt{1-4\alpha^2\rho}}}{\sqrt{2\alpha}}, \tag{19}$$

which become $s' \sim \pm\frac{1}{\sqrt{\alpha}}$ when $\alpha^2\rho \ll 1$. The resulting wave functions are $\psi(\rho) \sim \exp\left(\pm\frac{\rho}{\sqrt{\alpha}}\right)$. They are not physical states since they fail to satisfy the condition (13). One can also see that these two solutions are discarded according to the low-momentum consistency condition in [28]. In summary, assuming $\alpha \ll 1$, we find that the leading asymptotic behavior of the physically acceptable solution is $\exp\left(-\frac{2}{3}\rho^{\frac{3}{2}}\right)$ for $\rho \gg 1$. In addition, we only analyze the solution in the region $\alpha^2 |\rho| \ll 1$ where the GUP model is valid.

B. Integral Representation

The differential equation eqn. (12) can be solved by Laplace's method. Please refer to mathematical appendices of [29] for more details. Define the polynomials

$$P(t) = -\alpha^2 t^4 + t^2, \; Q(t) = -1, \tag{20}$$

and the function

$$Z(t) = \frac{1}{Q(t)} \exp\left(\int \frac{P(t)}{Q(t)} dt\right) = -\exp\left(\frac{\alpha^2 t^5}{5} - \frac{t^3}{3}\right). \tag{21}$$

Integral representations of the solutions to eqn. (12) are then given by

$$\begin{aligned} \psi(\rho) &= -\int_C \exp(\rho t) Z(t) \, dt \\ &= \int_C \exp\left(\rho t + \frac{\alpha^2 t^5}{5} - \frac{t^3}{3}\right) dt, \end{aligned} \tag{22}$$

where the contour C is chosen so that the integral is finite and non-zero and the function

$$V(t) = \exp\left(xt + \frac{\alpha^2 t^5}{5} - \frac{t^3}{3}\right), \tag{23}$$

vanishes at endpoints of C since the integrand of eqn. (22) is entire on the complex plane of t. Now that $\exp\left(xt+\frac{\alpha^2t^5}{5}-\frac{t^3}{3}\right)\sim\exp\left(\frac{\alpha^2t^5}{5}\right)$for large t, we need to begin and end the contour C in sectors for which $\cos 5\theta<0$ (setting $t=|t|e^{i\theta}$). There are five such sectors, specifically

$$\theta\in\Theta_n\equiv\left[\frac{2n\pi+\frac{\pi}{2}}{5},\frac{2n\pi+\frac{3\pi}{2}}{5}\right],\ n=0,1,2,3,4,5. \tag{24}$$

Therefore, any contour which originates at one of them and terminates at another yields a solution to eqn. (12). One could then find four linearly independent functions of the form

$$I_i\left(\rho\right)=\int_{C_i}\exp\left(\rho t+\frac{\alpha^2t^5}{5}-\frac{t^3}{3}\right)dt. \tag{25}$$

The asymptotic expression for $I_i\left(\rho\right)$ for large values of ρ is obtained by evaluating the integral eqn. (25) by the method of steepest descents.

III. ASYMPTOTIC EXPANSION

First we briefly review the method of steepest descent to introduce some useful formulas. This technique is very powerful to calculate integrals of the form

$$I\left(\rho\right)=\int_C g\left(z\right)e^{\rho f(z)}dz, \tag{26}$$

where C is a contour in the complex plane and $g\left(z\right)$ and $f\left(z\right)$ are analytic functions. The parameter ρ is real and we are usually interested in the behaviors of $I\left(\rho\right)$ as $\rho\to\pm\infty$. The key step of the method of steepest descent is applying Cauchy's theorem to deform the contours C to the contours consisting of steepest descent paths and other paths joining endpoints of two different steepest descent paths if necessary. Usually, the joining paths are chosen to make negligible contributions to $I\left(\rho\right)$. It is easy to show that $\operatorname{Im}f\left(z\right)$ is constant along steepest descent paths. When a steepest descent contour passes through a saddle point z_0 where $f'\left(z_0\right)=0$, $f\left(z\right)$ and $g\left(z\right)$ are expanded around z_0 and Watson's lemma is used to determine asymptotic behaviors of $I\left(\rho\right)$. Specifically, consider a contour C through a saddle point z_0. A new variable τ is introduced as $\tau=f\left(z\right)-f\left(z_0\right)$ to calculate $I\left(\rho\right)$. The saddle point z_0 divides the contour C into two contours C_1 and C_2. Generally, τ monotonically increases from $-\infty$ to zero along one contour, say C_1 and monotonically decreases from zero to $-\infty$ along C_2. Thus, the integral becomes

$$I\left(\rho\right)=\exp\left[\rho f\left(z_0\right)\right]\left[\int_{-\infty}^{0}g\left(\tau\right)\exp\left[\rho\tau\right]\frac{dz}{d\tau}|_{C_1}d\tau+\int_{0}^{-\infty}g\left(\tau\right)\exp\left[\rho\tau\right]\frac{dz}{d\tau}|_{C_2}d\tau\right]. \tag{27}$$

The physically acceptable solution can be represented by an integral

$$I(\rho) = \int_C \exp\left(\rho t + \frac{\alpha^2 t^5}{5} - \frac{t^3}{3}\right) dt, \tag{28}$$

where C is any contour which ranges from $t = \exp\left(-\frac{3\pi i}{5}\right)\infty$ to $t = \exp\left(\frac{3\pi i}{5}\right)\infty$. In fact, as we show later in the section for positive ρ, there exists a steepest descent contour from $t = \exp\left(-\frac{3\pi i}{5}\right)\infty$ to $t = \exp\left(\frac{3\pi i}{5}\right)\infty$, which C can be deformed to. Moreover, the integral on such a steepest descent contour yields the required asymptotic behavior of $I(\rho)$ at large positive value of ρ. Here the exponent in the integrand has movable saddle points. Making the change of variables $s = |\rho|^{\frac{1}{2}} t$, one gets

$$\begin{aligned} I(\rho) &= |\rho|^{\frac{1}{2}} \int_{\exp\left(-\frac{3\pi i}{5}\right)\infty}^{\exp\left(\frac{3\pi i}{5}\right)\infty} \exp\left[|\rho|^{\frac{3}{2}}\left(\pm s + \frac{as^5}{5} - \frac{s^3}{3}\right)\right] ds \\ &\equiv |\rho|^{\frac{1}{2}} \int_{\exp\left(-\frac{3\pi i}{5}\right)\infty}^{\exp\left(\frac{3\pi i}{5}\right)\infty} \exp\left[|\rho|^{\frac{3}{2}} f_{\pm}(s)\right] ds, \end{aligned} \tag{29}$$

where $+$ for $\rho > 0$ and $-$ for $\rho < 0$ and $a = \alpha^2 |\rho| \ll 1$ in the physical region.

A. Large Positive ρ

For $\rho > 0$, we have

$$f_+(s) = s + \frac{as^5}{5} - \frac{s^3}{3}. \tag{30}$$

There are four saddle points given by $f_+'(s) = 0$ at

$$s = \pm\lambda_+ \equiv \pm\frac{\sqrt{1-\sqrt{1-4a}}}{\sqrt{2a}} \text{ and } s = \pm\eta_+ \equiv \pm\frac{\sqrt{1+\sqrt{1-4a}}}{\sqrt{2a}}. \tag{31}$$

Our goal now is to find a steepest descent contour emerging from $s = \exp\left(-\frac{3\pi i}{5}\right)\infty$ to $s = \exp\left(\frac{3\pi i}{5}\right)\infty$. We will show that such a contour passes through $s = -\lambda_+$. To find the contour we substitute $s = u + iv$ and identify the real and imaginary parts of $f_+(s)$

$$\begin{aligned} f_+(s) = u &\left(1 - \frac{u^2}{3} + \frac{au^4}{5} + v^2 - 2au^2v^2 + av^4\right) \\ &+ iv\left(1 - u^2 + au^4 + \frac{v^2}{3} - 2au^2v^2 + \frac{av^4}{5}\right). \end{aligned} \tag{32}$$

Since $\operatorname{Im} f_+(-\lambda_+) = 0$, the constant-pahse contours passing through $s = -\lambda_+$ must satisfy

$$v\left(1 - u^2 + au^4 + \frac{v^2}{3} - 2au^2v^2 + \frac{av^4}{5}\right) = 0. \tag{33}$$

Therefore, one of the constant-phase contours passing through $s = -\lambda_+$ is

$$C : -\frac{1}{\sqrt{2a}}\sqrt{1 + 2av^2 - \sqrt{1 - 4a + \frac{8}{3}av^2 + \frac{16}{5}a^2v^4}} + iv, \text{ for} -\infty < v < \infty,$$

which is a steepest descent contour. In fact, around the saddle point $s = -\lambda_+$, one finds on the contour C

$$s \sim -\lambda_+ + bv^2 + iv, \tag{34}$$

and hence,

$$f_+(s) = f_+(-\lambda_+) - \frac{v^2}{2} f_+''(-\lambda_+) + \mathcal{O}\left(v^3\right), \tag{35}$$

where b is a positive real number. Since $f_+''(-\lambda_+)$ is real and positive, the contour C is indeed a steepest descent contour. Note that C goes to $s = \exp\left(-\frac{3\pi i}{5}\right)\infty$ as $v \to -\infty$ and $s = \exp\left(\frac{3\pi i}{5}\right)\infty$ as $v \to \infty$. In order to evaluate asymptotic expansion of $I(\rho)$, we break up the contour C into C_1 and C_2, corresponding to above and below of $s = -\lambda_+$. Define

$$\tau = f_+(s) - f_+(-\lambda_+), \tag{36}$$

where τ monotonically decreases from zero to $-\infty$ as one moves away from $s = -\lambda_+$ along C_1 to $s = \exp\left(\frac{3\pi i}{5}\right)\infty$ and along C_2 to $s = \exp\left(-\frac{3\pi i}{5}\right)\infty$, respectively. Since $f_+'(-\lambda_+) = 0$, the expression for s in terms of τ can be expressed as a power series of $\sqrt{-\tau}$. Then, noting that $-\tau = \left(\pm\sqrt{-\tau}\right)^2$, one has

$$s = -\lambda_+ + \sum_{j=1}^{\infty} a_j \left(\pm\sqrt{-\tau}\right)^j, \tag{37}$$

where a_i can be obtained by substituting eqn. (37) into eqn. (36) and equating powers of $\sqrt{-\tau}$ on both sides of the equations. It is easy to find

$$a_1 = i\sqrt{\frac{2}{|f_+''(-\lambda_+)|}}, \tag{38}$$

where one finds $\operatorname{Im} a_1 > 0$. The contour C_1 is in the second quadrant and hence, + sign is chosen in eqn. (37) for C_1. Therefore,

$$\begin{aligned}\rho^{\frac{1}{2}} \int_{C_1} \exp\left[\rho^{\frac{3}{2}} f_+(s)\right] ds &= \rho^{\frac{1}{2}} \exp\left[\rho^{\frac{3}{2}} f_+(-\lambda_+)\right] \int_0^{-\infty} \exp\left(\rho^{\frac{3}{2}}\tau\right) \frac{ds}{d\tau} d\tau \\ &\sim \exp\left[\rho^{\frac{3}{2}} f_+(-\lambda_+)\right] \sum_{j=1}^{\infty} \frac{j a_j}{2\rho^{\frac{3j-2}{4}}} \Gamma\left(\frac{j}{2}\right).\end{aligned} \tag{39}$$

For the contour segment C_2, the sign of $\sqrt{\tau}$ occurring in eqn. (37) has to be reversed. Moreover, the limit of integration on C_2 in the variable τ ranges from $-\infty$ to 0. Thus,

$$\rho^{\frac{1}{2}} \int_{C_1} \exp\left[\rho^{\frac{3}{2}} f_+(s)\right] ds \sim -\exp\left[\rho^{\frac{3}{2}} f_+(-\lambda_+)\right] \sum_{j=1}^{\infty} \frac{(-1)^j j a_j}{2\rho^{\frac{3j-2}{4}}} \Gamma\left(\frac{j}{2}\right). \tag{40}$$

Combining eqn. (39) and eqn. (40), we easily find

$$I\left(1 \ll \rho \ll \alpha^{-2}\right) \sim \frac{\exp\left[\rho^{\frac{3}{2}} f_+(-\lambda_+)\right]}{\rho^{\frac{1}{4}}} \sum_{j=0}^{\infty} \frac{(2j+1)\, a_{2j+1}}{\rho^{\frac{3j}{2}}} \Gamma\left(j+\frac{1}{2}\right). \tag{41}$$

B. Large Negative ρ

As for $\rho < 0$, the exponent in the integrand of $I(\rho)$ is

$$f_-(s) = -s + \frac{as^5}{5} - \frac{s^3}{3}. \tag{42}$$

Thus, one as well finds four saddle points given by $f'_+(s) = 0$

$$s = \pm\lambda_- \equiv \pm\frac{\sqrt{1-\sqrt{1+4a}}}{\sqrt{2a}} \text{ and } s = \pm\eta_- \equiv \pm\frac{\sqrt{1+\sqrt{1+4a}}}{\sqrt{2a}}. \tag{43}$$

As before, our objective is to find steepest descent contours passing through the saddle point(s) in eqn. (43) that emerges from $s = \exp\left(-\frac{3\pi i}{5}\right)\infty$ to $s = \exp\left(\frac{3\pi i}{5}\right)\infty$. Substituting $s = u + iv$, we obtain the real and imaginary parts of $f_-(s)$

$$\begin{aligned} f_-(s) = {} & u\left(-1 - \frac{u^2}{3} + \frac{au^4}{5} + v^2 - 2au^2v^2 + av^4\right) \\ & + iv\left(-1 - u^2 + au^4 + \frac{v^2}{3} - 2au^2v^2 + \frac{av^4}{5}\right). \end{aligned} \tag{44}$$

We have already shown that only one steepest descent contour passing through $s = -\lambda_+$ is sufficient to evaluate asymptotic behavior of $I(\rho)$ for large and positive ρ. However for large and negative ρ, things are a little bit more complicated. Instead of one steepest descent contour, it turns out that we need three steepest descent contours passing through $\pm\lambda_-$ and η_-, respectively, to connect two endpoints at $s = \exp\left(\pm\frac{3\pi i}{5}\right)\infty$.

First consider the steepest descent contour through $s = -\lambda_-$. Since $f_+(-\lambda_-)$ is a pure imaginary number, the steepest descent contour must satisfy

$$iv\left(-1 - u^2 + au^4 + \frac{v^2}{3} - 2au^2v^2 + \frac{av^4}{5}\right) = f_-(-\lambda_-). \tag{45}$$

Solutions to the last equation give us a constant phase contour $C_{-\lambda_-}$ passing through $s = -\lambda_-$, which emanates from $s = \exp\left(-\frac{3\pi i}{5}\right)\infty$ and finally approaches $s = \exp\left(-\frac{\pi i}{5}\right)\infty$. The contour $C_{-\lambda_-}$ actually is composed of three segments as

$$
\begin{aligned}
C_{-\lambda_-,1} &: -\frac{1}{\sqrt{2a}}\sqrt{1+2av^2-\sqrt{F_{-\lambda_-}(v)}}+iv, \text{ for } -\infty < v < -\operatorname{Im}\lambda_-, \\
C_{-\lambda_-,2} &: \frac{1}{\sqrt{2a}}\sqrt{1+2av^2-\sqrt{F_{-\lambda_-}(v)}}+iv, \text{ for } -\operatorname{Im}\lambda_- < v < -v_0, \\
C_{-\lambda_-,3} &: \frac{1}{\sqrt{2a}}\sqrt{1+2av^2+\sqrt{F_{-\lambda_-}(v)}}+iv, \text{ for } -v_0 > v > -\infty,
\end{aligned}
$$

where we define

$$
F_{\pm\lambda_-}(v) = 1 + 4a + \frac{8}{3}av^2 + \frac{16}{5}a^2v^4 + \frac{4af_-(\pm\lambda_-)}{iv},
$$

and v_0 is a solution to $F_{+\lambda_-}(v) = 0$ that satisfies $0 < v_0 \ll 1$. It is straightforward to verify that, along $C_{-\lambda_-}$, $\operatorname{Re} f_-(s)$ monotonically increases from $-\infty$ to 0 as one moves from $s = \exp\left(-\frac{3\pi i}{5}\right)\infty$ to $s = -\lambda_-$ and then monotonically decreases from 0 to $-\infty$ as one moves away from $s = -\lambda_-$ to $s = \exp\left(-\frac{\pi i}{5}\right)\infty$. Hence, the contour $C_{-\lambda_-}$ is indeed a the steepest descent contour passing through $s = -\lambda_-$. Now we calculate the contour integral on $C_{-\lambda_-}$. Introduce

$$
\tau = f_-(s) - f_-(-\lambda_-), \tag{46}
$$

which τ is real on $C_{-\lambda_-}$ and varies from $-\infty$ to zero and then to $-\infty$ along $C_{-\lambda_-}$. Then, one has

$$
s = -\lambda_- + \sum_{j=1}^{\infty} b_j \left(\pm\sqrt{-\tau}\right)^j, \tag{47}
$$

where b_i can be obtained by substituting eqn. (46) into eqn. (47). One easily gets

$$
b_1 = \exp\left(\frac{\pi}{4}i\right)\sqrt{\frac{2}{|f_-''(-\lambda_+)|}}. \tag{48}
$$

Since $\operatorname{Re}\exp\left(\frac{\pi}{4}i\right) > 0$, one has $-\sqrt{-\tau}$ for $C_{-\lambda_-,1}$ and $\sqrt{-\tau}$ for $C_{-\lambda_-,2} + C_{-\lambda_-,3}$ in eqn. (47). Therefore,

$$
\begin{aligned}
&|\rho|^{\frac{1}{2}} \int_{C_{-\lambda_-}} \exp\left[|\rho|^{\frac{3}{2}} f_-(s)\right] ds \\
&\sim 2\rho^{\frac{1}{2}} \exp\left[|\rho|^{\frac{3}{2}} f_-(-\lambda_-)\right] \int_0^{-\infty} \exp\left(|\rho|^{\frac{3}{2}}\tau\right) \sum_{j=0}^{\infty} (2j+1)\, b_j \left(\sqrt{-\tau}\right)^{2j} d\sqrt{-\tau} \\
&= \frac{\exp\left[|\rho|^{\frac{3}{2}} f_-(-\lambda_-)\right]}{|\rho|^{\frac{1}{4}}} \sum_{j=0}^{\infty} \frac{(2j+1)\, b_j}{|\rho|^{\frac{3j}{2}}} \Gamma\left(j+\frac{1}{2}\right).
\end{aligned} \tag{49}
$$

Analogously, one can readily write down a constant phase contour $C_{+\lambda_-}$ passing through $s=-\lambda_-$, which starts from $s=\exp\left(\frac{\pi i}{5}\right)\infty$ and ends at $s=\exp\left(\frac{3\pi i}{5}\right)\infty$. As before, $C_{+\lambda_-}$ consists of three segments

$$\begin{aligned}
C_{+\lambda_-,1} &: \frac{1}{\sqrt{2a}}\sqrt{1+2av^2+\sqrt{F_{+\lambda_-}(v)}}+iv, \text{ for } +\infty>v>v_0, \\
C_{+\lambda_-,2} &: \frac{1}{\sqrt{2a}}\sqrt{1+2av^2-\sqrt{F_{+\lambda_-}(v)}}+iv, \quad \text{for } v_0>v>\operatorname{Im}\lambda_-, \\
C_{+\lambda_-,3} &: -\frac{1}{\sqrt{2a}}\sqrt{1+2av^2-\sqrt{F_{+\lambda_-}(v)}}+iv, \text{ for } \operatorname{Im}\lambda_-<v<+\infty.
\end{aligned}$$

It is also straightforward to verify that $C_{+\lambda_-}$ is a steepest descent contour as well. Setting

$$\tau=f_-(s)-f_-(\lambda_-), \tag{50}$$

one finds τ is real on $C_{+\lambda_-}$ and varies from $-\infty$ to zero and then to $-\infty$ along $C_{+\lambda_-}$. Note that $f_+(s)$ is an odd function and $\lambda_-^*=-\lambda_-$. Taking complex conjugate of both sides of eqn. (50), one then has on $C_{+\lambda_-}$

$$s=\lambda_-+\sum_{j=1}^{\infty}b_j^*\left(\pm\sqrt{-\tau}\right)^j. \tag{51}$$

Since $\operatorname{Re}b_1^*>0$, one has $\sqrt{-\tau}$ for $C_{+\lambda_-,1}+C_{+\lambda_-,2}$ and $-\sqrt{-\tau}$ for $C_{+\lambda_-,3}$ in eqn. (51). Therefore,

$$|\rho|^{\frac{1}{2}}\int_{C_{+\lambda_-}}\exp\left[|\rho|^{\frac{3}{2}}f_-(s)\right]ds\sim-\frac{\exp\left[|\rho|^{\frac{3}{2}}f_-(\lambda_-)\right]}{|\rho|^{\frac{1}{4}}}\sum_{j=0}^{\infty}\frac{b_j^*(2j+1)}{|\rho|^{\frac{3j}{2}}}\Gamma\left(j+\frac{1}{2}\right). \tag{52}$$

Since the values of $\operatorname{Im}f_-(s)$ are different on $C_{\pm\lambda_-}$, it is obvious that we need a third contour which joins $C_{\pm\lambda_-}$ up at $s=\exp\left(\pm\frac{\pi i}{5}\right)\infty$, respectively. Here, we consider a constant phase contour C_{η_-} connecting $s=\exp\left(-\frac{\pi i}{5}\right)\infty$ to $\exp\left(\frac{\pi i}{5}\right)\infty$ that passes through η_-. Since $\operatorname{Im}f(s)=\operatorname{Im}f_-(\eta_-)=0$ on the contour C_{η_-}, one finds

$$C_{\eta_-}:\frac{1}{\sqrt{2a}}\sqrt{1+2av^2+\sqrt{1+4a+\frac{8}{3}av^2+\frac{16}{5}a^2v^4}}+iv, \text{ for } -\infty<v<+\infty,$$

is a curve of steepest descent. On C_{η_-}, define

$$\tau=f_-(s)-f_-(\eta_-), \tag{53}$$

which is real on C_{η_-} and varies from $-\infty$ to zero and then to $-\infty$ along C_{η_-}. Then, one finds

$$s = \lambda_- + \sum_{j=1}^{\infty} c_j \left(\pm\sqrt{-\tau}\right)^j, \tag{54}$$

and

$$c_1 = i\sqrt{\frac{2}{\left|f''_-\left(\eta_-\right)\right|}}. \tag{55}$$

Similarly, we break up the contour C_{η_-} into $C_{\eta_-,1}$ and $C_{\eta_-,2}$, corresponding to above and below of $s = \eta_-$ with $\sqrt{-\tau}$ for $C_{\eta_-,1}$ and $-\sqrt{-\tau}$ for $C_{\eta_-,2}$ in eqn. (54). Thus,

$$|\rho|^{\frac{1}{2}} \int_{C_{\eta_-}} \exp\left[|\rho|^{\frac{3}{2}} f_-\left(s\right)\right] ds \sim \frac{\exp\left[|\rho|^{\frac{3}{2}} f_-\left(\eta_-\right)\right]}{|\rho|^{\frac{1}{4}}} \sum_{j=0}^{\infty} \frac{\left(2j+1\right) c_{2j+1}}{|\rho|^{\frac{3j}{2}}} \Gamma\left(j + \frac{1}{2}\right). \tag{56}$$

Note that although paths C_{η_-} and $C_{\pm\lambda_-}$ never join up at $s = \exp\left(\pm\frac{\pi i}{5}\right)\infty$, the integrand $\exp\left[f_-\left(s\right)\right] \sim \exp\left(\frac{a|\rho|^{\frac{3}{2}}}{5} s^5\right)$ tends to zero exponentially. Therefore, there is no contribution from a connecting path from C_{η_-} and $C_{\pm\lambda_-}$ at a distance R from the origin in the limit $R \to \infty$. As a result, the integral $I\left(\rho\right)$ equals to the sum of three contour integrals on the different steepest descent curves C_{η_-} and $C_{\pm\lambda_-}$. Combining eqn. (49), eqn. (52) and eqn. (56) gives the full asymptotic expansion of $I\left(\rho\right)$ for large and negative ρ

$$\begin{aligned} I\left(-1 \gg \rho \gg -\alpha^{-2}\right) \sim\ & 2i \sum_{j=0}^{\infty} \frac{\operatorname{Im}\left(\exp\left[-|\rho|^{\frac{3}{2}} f_-\left(\lambda_-\right)\right] b_j\right)}{|\rho|^{\frac{1}{4}}} \frac{\left(2j+1\right)}{|\rho|^{\frac{3j}{2}}} \Gamma\left(j + \frac{1}{2}\right) \\ & + \frac{\exp\left[|\rho|^{\frac{3}{2}} f_-\left(\eta_-\right)\right]}{|\rho|^{\frac{1}{4}}} \sum_{j=0}^{\infty} \frac{\left(2j+1\right) c_{2j+1}}{|\rho|^{\frac{3j}{2}}} \Gamma\left(j + \frac{1}{2}\right). \end{aligned} \tag{57}$$

IV. WKB APPROXIMATION

The authors of [24] find the WKB approximation in deformed space with minimal length. In [24], they consider the deformed commutation relation

$$\left[X, P\right] = i\hbar f\left(P\right), \tag{58}$$

where $f\left(P\right)$ is an arbitrary function of P. In our paper, we set $f\left(P\right) = 1 + \beta P^2$. Defining $P\left(p\right)$

$$\frac{dP\left(p\right)}{dp} = f\left(P\right), \tag{59}$$

and $p(P)$ an inverse function of $P(p)$, they find the physical-optics approximation to the solution of the deformed Schrodinger equation

$$P\left(\frac{\hbar}{i}\frac{d}{dx}\right)\psi(x)+\frac{2m(E-V(x))}{\hbar^2}\psi(x)=0, \tag{60}$$

is

$$\psi(x)=\frac{1}{\sqrt{Pf(P)}}\left(C_1\exp\left[\frac{i}{\hbar}\int^x pdx\right]+C_2\exp\left[-\frac{i}{\hbar}\int^x pdx\right]\right), \tag{61}$$

where $P=\sqrt{2m(E-V(x))}$ in eqn. (61). It is also shown there that, if eqn. (61) is valid, the condition

$$\left|P^2\right|\gg\hbar\left|\frac{d}{dx}Pf(P)\right|, \tag{62}$$

has to be satisfied. However, the condition eqn. (62) fails near a turning point where $P=0$. Thus, if we want to determine bound state energies, we need to be able to match wave functions at the turning points. Here we considers a potential $V(x)$ with its classical turning point located at $x=0$. A linear approximation to the potential $V(x)$ near the turning point $x=0$ is

$$V(x)\approx V(0)+Fx, \tag{63}$$

where $F=V'(0)$. The linearized potential (63) is discussed in the previous two sections. Our discussion shows that the parameter $\alpha=\ell_\beta\left(2m|F|/\hbar^2\right)^{\frac{1}{3}}$ plays an important role in analyzing asymptotic behaviors of the solutions. When $\alpha\ll 1$, the physically acceptable solution can exist at large argument ρ while the condition (14) still holds. Accordingly, a turning points is called a smooth one if $\alpha=\ell_\beta\left(2m|F|/\hbar^2\right)^{\frac{1}{3}}\ll 1$. Otherwise, it is called a sharp turning point.

A. WKB Connection through a Smooth Turning Point

Now we want to match WKB wave functions at a smooth turning point in the deformed space with $f(P)=1+\beta P^2$ up to $\mathcal{O}(\beta)$. Suppose $x=0$ is a smooth turning point, which means $V(0)=E$, and $V>E$ for all $x>0$. The region to the left of the turning point is classically forbidden where the wave function must be damped and become zero at infinity. Thus, far from $x=0$, the wave function has the form

$$\psi(x)=\frac{1}{\sqrt{Pf(P)}}C\exp\left[-\frac{1}{\hbar}\left|\int_0^x pdx\right|\right],\quad\text{for } x>0. \tag{64}$$

To the right of the turning point, the wave function is given by

$$\psi(x)=\frac{1}{\sqrt{Pf(P)}}\left(C_1\exp\left[\frac{i}{\hbar}\int_0^x pdx\right]+C_2\exp\left[-\frac{i}{\hbar}\int_0^x pdx\right]\right),\text{ for } x<0. \tag{65}$$

Around the turning point, x is small and $P\sim\sqrt{2mF}\sqrt{-x}$. In this region, we may approximate eqn. (64) and eqn. (65) by

$$\psi(x)\approx(2mF\hbar)^{-\frac{1}{3}}x^{-\frac{1}{4}}\left(1+\frac{3a}{4}+\mathcal{O}\left(a^2\right)\right)C\exp\left[-\frac{2}{3\hbar}x^{\frac{3}{2}}\left(1+\frac{3}{10}a+\mathcal{O}\left(a^2\right)\right)\right],\text{ for } x>0, \tag{66}$$

$$\psi(x)=(2mF\hbar)^{-\frac{1}{3}}x^{-\frac{1}{4}}\left(1-\frac{3a}{4}+\mathcal{O}\left(a^2\right)\right)$$
$$\left(C_1\exp\left[\frac{2i}{3\hbar}|x|^{\frac{3}{2}}\left(1-\frac{3}{10}a+\mathcal{O}\left(a^2\right)\right)\right]+C_2\exp\left[-\frac{2i}{3\hbar}|x|^{\frac{3}{2}}\left(1-\frac{3}{10}a+\mathcal{O}\left(a^2\right)\right)\right]\right),\text{ for } x<0. \tag{67}$$

The criteria (62) for validity of the WKB approximation is satisfied if

$$|x|\gg\left(\frac{2mF}{\hbar^2}\right)^{-\frac{1}{3}}, \tag{68}$$

where we neglect βP^2 in derivation. On the other hand, when the potential is linearized around the turning point $x=0$, the Schrodinger equation becomes

$$\frac{d^2\psi(x)}{dx^2}-\ell_\beta^2\frac{d^4\psi(x)}{dx^4}-\frac{2m\mu x}{\hbar^2}\psi(x)\approx 0, \tag{69}$$

where $\beta=\frac{3\ell_\beta^2}{2\hbar^2}$. To solve the approximate differential equation, we make the substitution

$$\rho=x\left(2mF/\hbar^2\right)^{\frac{1}{3}}. \tag{70}$$

In terms of ρ, the solution to eqn. (69) which matches eqn. (66) and eqn. (67) in two different limits is actually $I(\rho)$ calculated in the section III. Specifically, the solution is

$$\psi(x)=DI(\rho)=DI\left(x\left(2mF/\hbar^2\right)^{\frac{1}{3}}\right), \tag{71}$$

where D is a constant to be determined by asymptotic matching. It is easily shown from (68) that there exists overlap regions where both WKB approximation and eqn. (69) hold. In the overlap regions, one finds $|\rho|\gg 1$ and $|x|\ll 1$. Therefore, we approximate $I(\rho)$ by its leading asymptotic behaviors for large argument in the the overlap regions. The appropriate

formulas are

$$I\left(1\ll\rho\ll\alpha^{-2}\right)\sim\frac{i\sqrt{\pi}\left(1+\frac{3a}{4}+\mathcal{O}\left(a^{2}\right)\right)}{\rho^{\frac{1}{4}}}\exp\left[-\frac{2\rho^{\frac{3}{2}}}{3}\left(1+\frac{3a}{10}+\mathcal{O}\left(a^{2}\right)\right)\right],\tag{72}$$

$$I\left(-1\gg\rho\gg-\alpha^{-2}\right)\sim\frac{2i\sqrt{\pi}\left(1-\frac{3a}{4}+\mathcal{O}\left(a^{2}\right)\right)}{|\rho|^{\frac{1}{4}}}\sin\left[\frac{2|\rho|^{\frac{3}{2}}}{3}\left(1-\frac{3a}{10}+\mathcal{O}\left(a^{2}\right)\right)+\frac{\pi}{4}\right],\tag{73}$$

where $\alpha\ll 1$ for a smooth turning point and $a=\ell_\beta^2\left(2mF/\hbar^2\right)^{\frac{2}{3}}|\rho|\ll 1$ as required by the condition (13). Requiring that eqn. (72) and eqn. (73) match eqn. (66) and eqn. (67) in the overlap region, respectively, gives $C_1=-iCe^{i\pi/4}$ and $C_2=iCe^{i\pi/4}$ up to $\mathcal{O}(\beta)$. In summary, in the overlap region, we find WKB solutions and the asymptotic values of the solution to the Schrodinger equation with a linear approximation to the potential $V(x)$. Then, by making eqn. (72) and eqn. (73) match eqn. (66) and eqn. (67) respectively, the WKB connection formula with the deformed commutator $[X,P]=i\hbar\left(1+\beta P^2\right)$ is obtained up to $\mathcal{O}(\beta)$. The connection formula around a smooth turning point is put in a way that

$$\frac{C}{\sqrt{Pf(P)}}\exp\left(-\frac{1}{\hbar}\left|\int_0^x pdx\right|\right)\to\frac{2C}{\sqrt{Pf(P)}}\sin\left(\frac{1}{\hbar}\int_0^x pdx+\frac{\pi}{4}\right),\text{ up to }\mathcal{O}(\beta),\tag{74}$$

which is directional, just as in ordinary quantum mechanics[27]. The analysis always proceeds from classically forbidden region to classically allowed one. For bound states, the uniqueness of the wave function in the classically allowed region leads to the Bohr-Sommerfeld quantization condition

$$\int_a^b pdx=\left(n+\frac{1}{2}\right)\pi\hbar,\text{ up to }\mathcal{O}(\beta),\tag{75}$$

where a and b are two smooth turning points for the potential $V(x)$. Notice that although eqn. (75) is claimed in [24], one still needs to obtain the connection formula to derive eqn. (75) rigorously, which is not presented in [24].

B. Discussion

1. *Sharp Turning Point*

Near a sharp turning point $x=0$, not only the WKB approximation falls apart but also matching the two WKB solutions across the turning point stops making sense. In fact, from

the previous subsection, one finds that the asymptotic matching is valid as long as there exists an overlap region where $1 \ll |\rho| \ll \alpha^{-2}$. However, such region doesn't exist unless $\alpha \ll 1$, which means that the asymptotic matching fails through a sharp turning point.

It can be shown, through (62), WKB approximations are valid as long as $|x| \gg \left(2m\left|F\right|/\hbar^2\right)^{-\frac{1}{3}}$ in the region where the potential is approximated by a linear one. Put another way, if there exists a region where both WKB and linear approximations are valid, one finds $|x| \gg \left(2m\left|F\right|/\hbar^2\right)^{-\frac{1}{3}}$ for such a region. When $|x| \gg \left(2m\left|F\right|/\hbar^2\right)^{-\frac{1}{3}}$, we have

$$\left|\beta P^2\right| \approx \frac{\ell_\beta^2}{\left(2m\left|F\right|/\hbar^2\right)^{\frac{2}{3}}} \frac{x}{\left(2m\left|F\right|/\hbar^2\right)^{\frac{1}{3}}} \gg 1 \tag{76}$$

for a sharp turning point. However, $|\beta P^2| \ll 1$ is required by the GUP model. This means that, as moving away from the sharp turning point, one is far beyond the region where the linear approximation to the potential is good before reaching the WKB valid region. One might resort to a higher order approximation to the potential and asymptotic matching in the overlap region to find WKB connection formula through a sharp turning point.

2. $\mathcal{O}(\beta)$ vs. $\mathcal{O}(\hbar)$

When $\hbar$ can be regarded as a small quantity, the approximate solution to the deformed Schrodinger equation

$$\frac{d^2\psi(x)}{dx^2} - \frac{2\hbar^2\beta}{3}\frac{d^4\psi(x)}{dx^4} + \frac{2m\left(E - V(x)\right)}{\hbar^2}\psi(x) = 0, \tag{77}$$

is easy to find using WKB analysis. To be specific, the approximate solution is expressed in an exponential power series of the form

$$\psi(x) \sim \exp\left[\frac{1}{\hbar}\sum_{n=0}^{\infty}\hbar^n S_n(x)\right]. \tag{78}$$

The authors of [24] finds

$$S_1 = -\frac{1}{2}\ln\left|2Pf(P)\right|. \tag{79}$$

Since here $f(P) = 1 + \beta P^2$, we have for S_1

$$S_1 \approx \ln\frac{1}{\sqrt{|P|}} - \frac{\beta}{2}P^2 + \mathcal{O}\left(\beta^2\right). \tag{80}$$

Moreover, the leading order (in terms of β) of the S_2 is just the WKB $\mathcal{O}(\hbar^2)$ correction calculated in the ordinary quantum mechanics. Therefore, we obtain[29]

$$S_2 \approx \frac{P'}{4P^2} + \int \frac{P'^2}{8P^3} dx + \mathcal{O}(\beta). \tag{81}$$

If one uses WKB approximations to evaluate quantum gravity induced corrections, say to energy levels or tunnelling rates, one may want to have

$$\beta P^2 \gtrsim \hbar S_2. \tag{82}$$

Otherwise, the quantum gravity correction ($\sim \mathcal{O}(\beta)$) on the first order WKB approximation ($\sim \mathcal{O}(\hbar^0)$) could be overwhelmed by the second order WKB approximation ($\sim \mathcal{O}(\hbar)$). Suppose a is the characteristic length of the potential $V(x)$, for example the width of a square-well potential. Then we can get a rough estimate on S_2

$$\hbar S_2 \sim \frac{\hbar}{aP} \sim \frac{\lambda}{a}, \tag{83}$$

where λ is the de Broglie wavelength of a particle with momentum P. As a result, the condition (82) becomes

$$\frac{\ell_\beta^2}{\lambda^2} \gtrsim \frac{\lambda}{a} \Rightarrow \lambda \lesssim \ell_\beta \left(\frac{a}{\ell_\beta}\right)^{\frac{1}{3}}. \tag{84}$$

It is interesting to note that the condition (84) is a rough estimate and a more accurate estimate could be obtained once the form of the potential is given.

Taking into account the constraints (84) on the de Broglie wavelength λ of a particle, one may conclude that the WKB approximation is not a powerful tool to calculate quantum gravity corrections unless the energy of the particle considered is high enough. However, there is an exception if the corresponding Schrodinger equation in the ordinary quantum mechanics can be solved exactly. In this case, $\mathcal{O}(\beta)$ corrections calculated on the WKB first order approximation are just quantum gravity corrections to exact results up to $\mathcal{O}(\beta)\mathcal{O}(\hbar^0)$ even without having (84) required. For example, if we employ WKB analysis to calculate the energy spectrum of a bound state in the deformed space, the energy levels can be represented by a series in powers of $\hbar$

$$E_n = \sum_{j=0}^{\infty} \hbar^j E_{n,j}(\beta), \tag{85}$$

where $E_{n,j}(\beta)$ can be expanded in terms of β

$$E_{n,j}(\beta) = \sum_{k=0}^{\infty} \beta^k E_{n,j}^k. \tag{86}$$

If on the first order WKB approximation, one calculates $E_{n,0}(\beta)$ up to $\mathcal{O}(\beta)$

$$E_{n,0}(\beta) = E_{n,0}^{0} + \beta E_{n,0}^{1} + \mathcal{O}\left(\beta^{2}\right), \tag{87}$$

the energy levels are

$$E_{n} = E_{n,0}^{0} + \beta E_{n,0}^{1} + \mathcal{O}\left(\beta^{2}\right) + \mathcal{O}(\hbar). \tag{88}$$

In order to have eqn. (88) make sense, one requires $\beta E_{n,0}^{1} \gtrsim \mathcal{O}(\hbar)$. On the other hand, if we know the exact result E_n with $\beta = 0$, namely $E_n^{(0)}$

$$E_{n}^{(0)} = \sum_{j=0}^{\infty} \hbar^{j} E_{n,j}^{0}, \tag{89}$$

eqn. (85) becomes

$$E_{n} = E_{n}^{(0)} + \beta E_{n,0}^{1} + \mathcal{O}(\hbar)\,\mathcal{O}(\beta) + \mathcal{O}\left(\beta^{2}\right). \tag{90}$$

Since $\mathcal{O}(\hbar)\,\mathcal{O}(\beta)$ is automatically smaller than βE_{0}^{1}, eqn. (90) always makes sense as long as $\mathcal{O}(\hbar) \ll 1$.

To illustrate our points, we use the WKB approximation to derive the energy levels of a particle confined to the one-dimensional potential $V(x) = F|x|$ whose turning points are

$$a = -\frac{E}{F},\ b = \frac{E}{F}. \tag{91}$$

The energy quantization condition (75) from first order WKB approximation then becomes

$$\ell_{F}^{-\frac{3}{2}} \int_{-\frac{E}{F}}^{\frac{E}{F}} \sqrt{\frac{E}{F} - |x|}\,dx - \frac{\ell_{\beta}^{2}}{2\ell_{F}^{\frac{9}{2}}} \int_{-\frac{E}{F}}^{\frac{E}{F}} \left(\frac{E}{F} - |x|\right)^{\frac{3}{2}} dx = \left(n + \frac{1}{2}\right)\pi + \mathcal{O}\left(\beta^{2}\right), \tag{92}$$

where $\ell_F = \left(\hbar^2/2mF\right)^{\frac{1}{3}}$ is the characteristic length of the potential $V(x) = F|x|$. From the last equation, we obtain

$$\frac{E_{n}}{F} \approx \ell_{n}\left(1 + \frac{\ell_{\beta}^{2}\ell_{n}}{5\ell_{F}^{3}} + \mathcal{O}\left(\beta^{2}\right) + \mathcal{O}(\hbar)\right), \tag{93}$$

where $\ell_n = \ell_F \left[\frac{3}{4}\left(n + \frac{1}{2}\right)\pi\right]^{\frac{2}{3}}$. What is $\mathcal{O}(\hbar)$? The second order generalization of eqn. (75) with $\beta = 0$ is given in [27]

$$\ell_{F}^{-\frac{3}{2}} \int_{-\frac{E^{(0)}}{F}}^{\frac{E^{(0)}}{F}} \sqrt{\frac{E^{(0)}}{F} - |x|}\,dx + \frac{F_{F}^{\frac{3}{2}}\ell^{\frac{3}{2}}}{48E^{(0)\frac{3}{2}}} = \left(n + \frac{1}{2}\right)\pi + \mathcal{O}\left(\hbar^{2}\right), \tag{94}$$

which gives

$$\frac{E_{n}^{(0)}}{F} \approx \ell_{n}\left(1 - \frac{\ell_{F}^{3}}{96\ell_{n}^{3}} + \mathcal{O}\left(\hbar^{2}\right)\right). \tag{95}$$

We can then estimate $\mathcal{O}(\hbar)$ through (95)

$$\mathcal{O}(\hbar) \sim \frac{\ell_F^3}{\ell_n^3}, \tag{96}$$

which can also be easily obtained by dimensional analysis. If one wants the first order approximation (93) to make sense, the second term in (93) should be comparable to or larger than $\mathcal{O}(\hbar)$ and then one gets

$$\ell_n \gtrsim \ell_F \sqrt{\frac{\ell_F}{\ell_\beta}}. \tag{97}$$

The de Broglie wavelength of a particle with energy $E_n \sim F\ell_n$ is

$$\lambda_n \sim \frac{\hbar}{\sqrt{2mF\ell_n}} \sim \frac{\ell_F^{\frac{3}{2}}}{\sqrt{\ell_n}}. \tag{98}$$

Thus, the inequality (97) reads

$$\lambda_n \lesssim \ell_F \left(\frac{\ell_\beta}{\ell_F}\right)^{\frac{1}{4}}, \tag{99}$$

which is much milder than (84). In a practical way, $\hbar$ and β can be expressed in terms of ℓ_β, l_F and l_n. In fact, it is easily shown that

$$\mathcal{O}(\hbar^m) \sim \mathcal{O}\left(\frac{\ell_F^{3m}}{\ell_n^{3m}}\right) \sim \mathcal{O}\left(\frac{1}{n^{2m}}\right), \quad \mathcal{O}(\beta^m) \sim \mathcal{O}\left(\frac{\ell_\beta^{2m}}{\ell_F^{2m}}\right). \tag{100}$$

V. CONCLUSIONS

In this paper, we considered a homogeneous field in the deformed quantum mechanics with minimal length. The physical motivation for this is to obtain the WKB connection formula and prove the Bohr-Sommerfeld quantization rule rigorously in the deformed quantum mechanics. By studying the leading asymptotic behavior of the physically acceptable wave function in the physical region, we found the contour for its integral representation. Through the integral representation, the asymptotic expansions of the physically acceptable wave function at both large positive and large negative values of ρ were given.

Then, we used the obtained asymptotic expansions to get the WKB connection formula, which proceeds from classically forbidden region to classically allowed one through a smooth turning point, and had the Bohr-Sommerfeld quantization rule proved rigorously up to $\mathcal{O}(\beta)$. A new interesting feature appearing in the presence of deformation is that our WKB

connection formula do not work for a sharp turning point. The connection through such a point might need a higher order approximation to the potential near it.

Finally, we discussed the competition between the quantum gravity correction on the first order WKB approximation and the second order WKB approximation. If the former is not overwhelmed by the latter, the energy of the particle considered should be high enough according to (84). We also showed that, if the energy levels $E_n^{(0)}$ of a bound state are given in the ordinary quantum mechanics, the deformed energy levels are

$$E_n = E_n^{(0)} + \beta E_{n,0}^1 + \mathcal{O}(\hbar)\,\mathcal{O}(\beta) + \mathcal{O}\left(\beta^2\right), \tag{101}$$

where $\beta E_{n,0}^1$ is the $\mathcal{O}(\beta)$ quantum gravity correction on the first order WKB approximation.

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[1] G. Veneziano, *A stringy nature needs just two constants*, Europhys. Lett. **2** (1986) 199.

[2] D. J. Gross and P. F. Mende, *String theory beyond the Planck scale*, Nucl. Phys. **B 303** (1988) 407.

[3] D. Amati, M. Ciafaloni and G. Veneziano, *Can space-time be probed below the string size?* Phys. Lett. **B 216** (1989) 41.

[4] K. Konishi, G. Paffuti and P. Provero, *Minimum physical length and the generalized uncertainty principle in string theory*, Phys. Lett. **B 234** (1990) 276.

[5] R. Guida, K. Konishi and P. Provero, *On the short distance behavior of string theories*, Mod. Phys. Lett. **A 6** (1991) 1487.

[6] M. Maggiore, *A generalized uncertainty principle in quantum gravity*, Phys. Lett. **B 304** (1993) 65.

[7] M. Maggiore, *The algebraic structure of the generalized uncertainty principle*, Phys. Lett. **B 319** (1993) 83.

[8] L. J. Garay, *Quantum gravity and minimum length*, Int. J. Mod. Phys. **A 10** (1995) 145.

[9] A. Kempf, G. Mangano and R. B. Mann, *Hilbert space representation of the minimal length uncertainty relation*, Phys. Rev. **D 52** (1995) 1108.

[10] A. Kempf, *Quantum group-symmetric fock spaces with bargmann-fock representation*, Lett. Math. Phys. **26** (1992) 1.

[11] A. Kempf, *Quantum group Symmetric Bargmann-Fock space: integral kernels, Green functions, driving forces*, J. Math. Phys. **34** (1994) 969.

[12] A. Kempf, *Uncertainty relation in quantum mechanics with quantum group symmetry*, J. Math. Phys. **35** (1994) 4483.

[13] S. Hossenfelder, *A note on theories with a minimal length*, Class. Quantum Grav. **23** (2006) 1815.

[14] S. Hossenfelder, *Minimal Length Scale Scenarios for Quantum Gravity,* arXiv:1203.6191v1.

[15] S. Das and E.C. Vagenas, *Universality of Quantum Gravity Corrections*, Phys. Rev. Lett. 101 (2008) 221301.

[16] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, *Exact Solution of the Harmonic Oscillator in Arbitrary Dimensions with Minimal Length Uncertainty Relations*, Phys. Rev. D 65 (2002) 125027.

[17] R. Akhoury and Y.-P. Yao, *Minimal Length Uncertainty Relation and the Hydrogen Spectrum*, Phys. Lett. B 572 (2003) 37.

[18] T. V. Fityo, I. O. Vakarchuk and V. M. Tkachuk, *One dimensional Coulomb-like problem in deformed space with minimal length*, J. Phys. A: Math. Gen. 39 (2006) 2143.

[19] F. Brau, and F. Buisseret, *Minimal Length Uncertainty Relation and gravitational quantum well*, Phys. Rev. D 74 (2006) 036002.

[20] P. Pedram, *Minimal Length and the Quantum Bouncer: A Nonperturbative Study*, Int. J. Theor. Phys. 51 (2012) 1901.

[21] A. F. Ali, S. Das and E. C. Vagenas, *Discreteness of space from the generalized uncertainty principle*, Phys. Lett. B 678 (2009) 497.

[22] A. F. Ali, S. Das and E. C. Vagenas, *A proposal for testing Quantum Gravity in the lab*, Phys. Rev. D 84 (2011) 044013.

[23] P. Pedram, *New Approach to Nonperturbative Quantum Mechanics with Minimal Length Uncertainty*, Phys. Rev. D 85 (2012) 024016.

[24] T.V. Fityo, I.O. Vakarchuk and V.M. Tkachuk, *WKB approximation in deformed space with*

minimal length, J. Phys. A: Math. Gen. 39 (2006) 379.

[25] P. Pedram, *A Higher Order GUP with Minimal Length Uncertainty and Maximal Momentum*, Phys. Lett. B 714 (2012) 317.

[26] C. L. Ching, R. Parwani and K. Singh, *The Constraints and Spectra of a Deformed Quantum Mechanics*, arXiv:1204.1642.

[27] C. M. Bender and S. A. Orszag, *Applied Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978).

[28] C. L. Ching and R. Parwani, *Scattering and Bound States of a Deformed Quantum Mechanics*, arXiv:1207.1519.

[29] L. Landau and E. Lifshitz, *Quantum Mechanics*, Oxford (2000).